ON A NEW ITERATIVE ALGORITHM
FOR FINDING THE SOLUTIONS OF GAMES
AND LINEAR PROGRAMMING PROBLEMS

Richard Bellman

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ON A NEW ITERATIVE ALGORITHM FOR FINDING THE SOLUTIONS OF GAMES AND LINEAR PROGRAMMING PROBLEMS

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§1. Introduction.

The purpose of this paper is to present a new iterative procedure for finding the solutions of games. Since, as is now well known, every linear programming problem of conventional type can be transformed into a symmetric game, this technique may be applied to the solution of linear programming problems. It was actually with this purpose in mind that the method was developed. Our aim is to obtain a procedure which converges more rapidly than either the statistical method of Brown, or the differential equation approach of Brown and von Neumann.

We shall first present a variant of the differential equation approach which converges more rapidly than the original. Carrying this approach to its logical limit we obtain a process with an exponential rate of convergence. The discrete analogue, obtained by replacing the differential equation by a difference equation, furnishes the new iterative algorithm.

In replacing the differential equation by a difference equation, the expression dx/dt is replaced by $(x(t+h)-x(t))/h$ with t taking the values 0, h, $2h$, etc. At the N^{th} step the error will be proportional to e^{-Nh}. Since, in general, h must be taken small to insure that the solution of the difference equation is close to

the solution of the differential equation, a large number of iterations will be required to obtain an accurate approximation.

To counteract this unpleasant feature we have devised a combination of a continuous and discrete iteration process which seems well suited to hand computation. For games of fairly large size it would seem that the most economic policy, taking account of the cost of both hand and machine computation, would be to use the continuous iteration technique up to a certain point and then let the machines take over using a discrete technique.

The method has been tried on a number of examples, and in all cases it seems to yield the value of the game very rapidly. It is conceivable that in certain linear programming problems it might be desired only to determine the value of the maximum or minimum rather than the variables which yield the critical value. For problems of this type the method seems admirably suited.

I would like to express my sincere appreciation to Bernice Brown, without whose enthusiastic interest and cooperation the method would have perished still-born.

§2. The Brown-von Neumann Technique.

Let us consider a symmetric game characterized by the skew-symmetric matrix (a_{ij}) and set

$$(1)\quad \begin{array}{ll} \text{(a)} & x_i \geq 0, \quad \sum_{i=1}^{N} x_i = 1 \\ \text{(b)} & u_i = \sum_{j=1}^{N} a_{ij}x_j \\ \text{(c)} & \phi(u_i) = \text{Max}\,[0,u_i] \\ \text{(d)} & \phi(x) = \sum_i \phi(u_i) \end{array}$$

The system of differential equations to be used to yield a solution is

$$(2) \qquad \frac{dx_i}{dt} = \phi(u_i) - \phi(x)x_i,$$

$$x_i(0) = c_i,$$

with $c_i \geq 0$, $\sum_{i=1}^{N} c_i = 1$.

An ingenious argument from this point on shows that $\phi(u_i) \longrightarrow 0$ as $t \longrightarrow \infty$, which means that the $x_i(t)$ must have cluster points which furnish solutions to the game. These may be obtained by inspection of the numerical results.

The convergence of $\phi(u_i)$ is of the order of $1/t$ as $t \longrightarrow \infty$.

§3. A Variant of the Brown-von Neumann Technique.

In an effort to speed up convergence, let us see what happens if we replace (2) of §2 by

$$(1) \qquad \frac{dx_i}{dt} = \phi(u_i)^{\alpha} - \phi_{\alpha}(x)x_i, \quad \alpha > 0,$$

where

$$(2) \qquad \phi_{\alpha}(x) = \sum_{i=1}^{N} \phi(u_i)^{\alpha}.$$

We need no longer have uniqueness of solution if $0 < \alpha < 1$. This, however, is no handicap since the method will show that any solution of (1) will have the property that $\phi(u_i) \longrightarrow 0$.

Repeating the Brown-von Neumann argument we have, if $\phi(u_i) > 0$,

$$(3) \qquad \frac{d}{dt}\phi(u_i) = \sum_j a_{ij}\frac{dx_j}{dt} = \sum_j a_{ij}\left[\phi(u_j)^\alpha - \phi_\alpha(x)x_j\right]$$

$$= \sum_j a_{ij}\,\phi(u_j)^\alpha - \phi_\alpha(x)\sum_j a_{ij}x_j.$$

Hence for all t,

$$(4) \qquad \phi(u_i)^\alpha \frac{d}{dt}\phi(u_i) = \sum_j a_{ij}\phi(u_j)^\alpha\,\phi(u_i)^\alpha$$

$$- \phi_\alpha(x)\,\phi(u_i)^{\alpha+1}.$$

Summing over i this yields

$$(5) \qquad \frac{1}{1+\alpha}\,\frac{d}{dt}\left(\sum_i \phi(u_i)^{\alpha+1}\right) = -\phi_\alpha(x)\sum_i \phi(u_i)^{\alpha+1},$$

since

$$(6) \qquad \sum_{i,j} a_{ij}\,\phi(u_j)^\alpha\,\phi(u_i)^\alpha = 0$$

by virtue of the skew symmetry $a_{ij} = -a_{ji}$.

Let us now use the fact that

$$(7) \qquad \phi_\alpha(x) = \sum_i \phi(u_i)^\alpha \geq k(\alpha)\left(\sum_i \phi(u_i)^{\alpha+1}\right)^{\alpha/\alpha+1},$$

where $k(\alpha)$ is an appropriate constant. Hence

$$(8) \qquad \frac{1}{(1+\alpha)} \frac{d}{dt} \Big(\sum_i \phi(u_i)^{\alpha+1} \Big) < -k(\alpha) \Big(\sum_i \phi(u_i)^{\alpha+1} \Big)^{1+\frac{\alpha}{\alpha+1}} .$$

Writing $v = \sum_i \phi(u_i)^{\alpha+1}$, (8) becomes

$$(9) \qquad \frac{dv}{dt} < -k_1 v^{1+\frac{\alpha}{\alpha+1}} .$$

Integrating this inequality we obtain

$$(10) \qquad v \leq \frac{1}{(k_1 t + k_2)^{(1+\alpha)/\alpha}}$$

where k_1 and k_2 are positive constants.

The interesting feature about this result is that the smaller α is chosen, the more rapid the convergence.

§4. <u>The Limiting Case</u> $\alpha = 0$.

The above observation leads us to consider the case $\alpha = 0$. The differential equation takes the form

$$(1) \qquad \frac{dx_i}{dt} = f(u_i) - \psi(x)x_i,$$

where

$$(2) \qquad f(u_i) = 1 \text{ if } u_i > 0,$$

$$= 0 \text{ if } u_i \leq 0,$$

$$\psi(x) = \sum_{i=1}^{N} f(u_i).$$

We have

$$(3) \qquad \frac{du_i}{dt} = \sum_{j=1}^{N} a_{ij} \frac{dx_j}{dt} = \sum_{j=1}^{N} a_{ij} \left[f(u_j) - \psi(x) x_j \right]$$

$$= \sum_{j=1}^{N} a_{ij} f(u_j) - \psi(x) u_i ,$$

and, as above,

$$(4) \qquad \sum_{i=1}^{N} f(u_i) \frac{du_i}{dt} = - \psi(x) \sum_{i=1}^{N} u_i f(u_i)$$

Hence

$$(9) \qquad \frac{d}{dt} \Big(\sum_{i=1}^{N} u_i f(u_i) \Big) = - \psi(x) \sum_{i=1}^{N} u_i f(u_i) ,$$

which yields

$$(10) \qquad \sum_{i=1}^{N} u_i f(u_i) = a_1 e^{- \int_0^t \psi(x) dt_1} .$$

Since $\psi(x) \geq 1$ as long as the system is not in its equilibrium position we see that

$$(11) \qquad \sum_{i=1}^{N} u_i f(u_i) \leq a_1 e^{-t} ,$$

as long as one f is positive. This shows that the convergence is exponential.

§5. The Difference Equation.

For computing purposes we write (1) of §4 in the form

$$(1) \qquad x_i(n+1) - x_i(n) = h\left[f(u_i(n)) - \phi(x(n))x_i(n)\right],$$

or

$$(2) \qquad x_i(n+1) = x_i(n)(1-h\phi(x(n)) + hf(u_i(n)).$$

In this form, although $\sum_i x_i(n)$ is equal to 1 for all n, an individual x_i may go negative to a small degree.

To avoid this, we replace (1) by

$$(3) \qquad x_i(n+1) - x_i(n) = h\left[f(u_i(n)) - \phi(x(n))x_i(n+1)\right],$$

which yields

$$(4) \qquad x_i(n+1) = \frac{x_i(n) + hf(u_i(n))}{1 + h\phi(x(n))}.$$

It is probably true that Max $[0, u_i(n)]$ goes to zero as $n \longrightarrow \infty$, and exponentially fast, although we have not proved this. Computation on particular games with small values of h would seem to bear this out. A discussion of some of these will be given subsequently.

§6. A Procedure for Speeding up the Discrete Iteration.

Since $f(u_i(n))$, and hence $\phi(x(n))$, depend only upon the sign of $u_i(n)$, not its magnitude, it is clear that the iteration formula will be the same for long stretches. Furthermore, if h is small, (4)

of §5 may be replaced by (2).

If one step of the iteration is performed, linear interpolation method can now be used to continue the iteration until a boundary is reached, that is to say, a point where some $\sum a_{ij}x_j$ changes sign.

In this way, performing the equivalent of one hundred or more steps at a time the hand computer can, in the early stages of the computation, keep pace with and perhaps even gain on a machine. When, however, the values are close to an actual solution the sign changes become so frequent that a machine is more efficient.

§7. A 7 x 7 Matrix.

Let us consider the following game matrix

$$(1) \qquad A = \begin{pmatrix} 0 & -1 & -2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 4 & 0 & 0 & -1 \\ 2 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & -4 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & -3 & 1 \\ -2 & 0 & 0 & 1 & 3 & 0 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

which has the unique solution

$$(2) \qquad x_1 = .111, \quad x_2 = .111, \quad x_3 = .167, \quad x_4 = .056,$$
$$x_5 = .167, \quad x_6 = .056, \quad x_7 = .333$$

where the value of the original 3 x 3 game which gave rise to this skew-symmetric matrix is 1, the last component.

To begin with we take h = .001 and make an initial guess

(3) $x_1 = x_2 = \cdots x_7 = 1/7$

Following we present the first two pages of the calculation and the subsequent values of x_7.

-473

	x(0)	Σax	x(1)	Σax	x(2)	Σax	x(3)	Σax
1	.1429	0	1425	-0010	1419	-0010	1413	0010
2	.1429	5714	1435	5700	1439	5666	1443	5632
3	.1429	1429	1435	1425	1439	1409	1443	1393
4	.1429	-4286	1425	-4315	1419	-4327	1413	-4339
5	.1429	-4286	1425	-4315	1419	-4327	1413	-4339
6	.1429	1429	1435	1425	1439	1399	1443	1393
7	.1429	0	1425	0030	1429	0060	1433	0090

	x(90)	Σax	x(91)	Σax	x(95)	Σax	x(96)	Σax
1	.0926	-0005	0922	-0005	0907	-0005	0905	-0005
2	.1808	2827	1811	2804	1822	2718	1828	2702
3	.1808	0049	1811	0038	1822	-0003	1818	-0013
4	.0926	-5429	0922	-5438	0907	-5471	0905	-5489.
5	.0926	-5429	0922	-5438	0907	-5471	0905	-5449
6	.1808	0049	1811	0038	1822	-0003	1818	-0013
7	.1803	2646	1806	2667	1817	2745	1823	2749

	x(265)	Σax	x(266)	Σax	x(278)	Σax	x(279)	Σax
1	.0601	-0005	0600		0586	-0005	0586	-0005
2	.2893	0117	2897	0108	2947	-0012	2944	-0019
3	.1203	-1686	1201		1172	-1770	1171	-1777
4	.0601	-8684	0600		0586	-8846	0586	-8827
5	.0601	-1924	0600		0586	-1746	0586	-1735
6	.1203	-1686	1201		1172	-1770	1171	-1777
7	.2888	3496	2892		2942	3533	2949	3528

	x(329)	Σax	x(330)	Σax	x(331)	Σax	x(431)	Σax
1	.0556	0500	0565	0509	0574	0518	1454	1418
2	.2799	-0519	2793	-0517	2787	-0515	2227	-0415
3	.1111	-2187	1109	-2172	1107	-2157	0887	-0737
4	.0556	-7897	0555	-7870	0554	-7843	0444	-5263
5	.0556	-1145	0555	-1134	0554	-1123	0444	0097
6	.1111	-2187	1109	-2212	1107	-2237	0887	-4777
7	.3299	3353	3302	3336	3305	3319	3645	1659

	x(432)	Σax	x(433)	Σax	x(464)	Σax	x(465)	Σax
1	.1460	1424	1466	1430	1640		1645	1609
2	.2220	–0412	2213	–0409	2008		2002	–0362
3	.0884	–0714	0881	–0691	0800	–0005	0798	0015
4	.0443	–5236	0442	–5209	0402		0401	–4397
5	.0453	0108	0462	0119	0729		0737	0419
6	.0884	–3302	0881	–4747	0800		0798	–4289
7	.3644	1632	3643	1605	3612		3611	0815

	x(466)	Σax	x(467)	Σax	x(533)	Σax	x(534)	Σax	x(536)	Σax
1	.1648	1593	1651		1875	0522	1879	0500	1887	0459
2	.1994	–0383	1986		1458	–0287	1454	–0277	1445	–0257
3	.0805	0034	0812		1254	1346	1260	1371	1272	1421
4	.0399	–4359	0397		0291	–1833	0290	–1819	0288	–1788
5	.0744	0417	0751	0415	1213	0329	1219	0208	1231	0296
6	.0795	–4272	0792		0581	–3146	0579	–3127	0576	–3089
7	.3607	0803	3603	0791	3326	–0086	3316	–0095	3296	–0113

	x(549)	Σax	x(559)	Σax	x(560)	Σax	x(561)	Σax	x(562)	Σax
1	.1943	0179	1984	–0026	1980		1976		1972	
2	.1389	–0119	1347	–0015	1344	–0017	1341	–0019	1338	
3	.1353	1757	1412	2002	1419		1426		1433	
4	.0276	–1591	0268	–1442	0267		0266		0265	
5	.1313	0151	1373	0048	1380	0038	1387	0028	1394	0018
6	.0554	–2867	0537	–2652	0536		0535		0534	
7	.3166	–0256	3071	–0329	3065		3059		3053	

	x(563)	Σax	x(564)	Σax	x(565)	Σax
1	.1968		1964	–0121	1962	–0143
2	.1335		1332	–0025	1331	–0024
3	.1440		1447	2032	1456	2090
4	.0264		0263	–1372	0263	–1361
5	.1401	0008	1408	–0002	1407	–0011
6	.0533		0532	–2482	0531	–2298
7	.3047		3041	–0324	3038	–0314

(4)

x_7	N	$\sum_i \text{Max} (\Sigma a_{ij} x_j, 0)$
.3008	561	.203 +
.2944	593	.197 +
.3227	743	.13 +
.3539	830	.11 +
.3077	886	.10 +

§8. A 21 x 21 Matrix.

With the aim of testing the convergence of the value, we obtained the following 21 x 21 matrix with value .5 from Ruth Wagner.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1											2	3	2	2	2	1	3	3	3	2	-1
2											1	2	2	2	3	2	1	2	2	3	-1
3											2	2	2	3	2	2	3	2	3	2	-1
4											2	2	1	2	2	2	1	2	2	3	-1
5					0						2	1	2	2	2	3	3	2	1	2	-1
6											3	2	2	2	1	2	1	2	2	2	-1
7											1	3	1	3	1	3	2	3	3	1	-1
8											1	2	2	2	2	2	1	2	2	2	-1
9											1	2	1	2	3	2	1	2	2	3	-1
10											2	1	2	1	2	2	3	2	1	2	-1
11	-2	-1	-2	-2	-2	-3	-1	-1	-1	-2											1
12	-3	-2	-2	-2	-1	-2	-3	-2	-2	-1											1
13	-2	-2	-2	-1	-2	-2	-1	-2	-1	-2											1
14	-2	-2	-3	-2	-2	-2	-3	-2	-2	-1											1
15	-2	-3	-2	-2	-2	-1	-1	-2	-3	-2											1
16	-1	-2	-2	-2	-3	-2	-3	-2	-2	-2						0					1
17	-3	-1	-3	-1	-3	-1	-2	-1	-1	-3											1
18	-3	-2	-2	-2	-2	-2	-3	-2	-2	-2											1
19	-3	-2	-3	-2	-1	-2	-3	-2	-2	-1											1
20	-2	-3	-2	-3	-2	-2	-1	-2	-3	-2											1
21	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0

Following are the results of the computations. Observe that the convergence of x_{21} to the neighborhood of .5 is extremely rapid.

	x(0)	Σax	x(1)	Σax	x(91)	Σax	x(191)	Σax
1	.047619	1.047619	.048138	1.036243	.090020	–.067481	–.081020	–.160678
2	.047619	.904761		.894802		–.070418		–.163316
3	.047619	1.047619		1.036243		–.067481		–.160678
4		.857142		.847655		–.071397		–.164194
5		.904761		.894802		–.070418		–.163316
6		.857142		.847655		–.071397		–.164194
7		.952381		.941949		–.069439		–.162437
8		.809523		.800508		–.072376		–.165074
9		.857142		.847655		–.071397		–.164194
10		.809523		.800508	.090020	–.072376	–.081020	–.165074
11		–.761904	.047147	–.771198	.000979	–1.440340	.000879	–1.196440
12		–.904761		–.915613		–1.710400		–1.439500
13		–.761904		–.771198		–1.440340		–1.196440
14		–.952381		–.963751		–1.800420		–1.520520
15		–.904761		–.915613		–1.710400		–1.439500
16		–.952381		–.963751		–1.800420		–1.520520
17		–.857142		–.867475		–1.620380		–1.358480
18		–1.000000	.047147	–1.011889		–1.890440		–1.601540
19		–.952381	.047148	–.963751	.000980	–1.800420	.000880	–1.520520
20		–1.000000	.047148	–1.011889	.000980	–1.890440		–1.601540
21		–0	.047148	.009910	.090000	.890408	.180900	.801408

	x(691)	Σax	x(741)	Σax	x(791)	Σax	x(841)	Σax
1	.040520	−.581178	.038494	−.602088	.036569	−.621934	.031084	−.399084
2								
3								
4								
5								
6								
7								
8								
9								
10	.040520		.038494		.036569		.031084	
11	.000379	−.098940	.000359	−.044048	.000341	.008109	.045141	.056754
12						−.101598	.000291	−.036498
13						.008109	.045141	.056754
14						−.138167	.000291	−.067582
15						−.101598	.000291	−.036498
16						−.138167	.000291	−.067582
17				−.121036		−.065029	.000291	−.005414
18						−.174736	.000291	−.098666
19	.000380		.000360		.000342	−.138167	.000292	−.067582
20	.000380		.000360		.000342	−.174736	.000292	−.098666
21	.589900	.408992	.610350	.381348	.629782	.369102	.585182	.218228

	x(861)	Σax	x(871)	Σax	x(881)	Σax	x(891)	Σax	x(892)
1	.028607	-.247762	.027369	-.172109	.025737	-.052497	.024105	.062809	.024856
2						-.204469		-.111237	.023866
3						-.062437		.043521	.024856
4						-.214646		-.130751	.023866
5						-.073102		.023780	.024856
6						-.074245		.011912	.024856
7						-.251304		-.165080	.023866
8						-.214892		-.130978	.023866
9						-.279883		-.192439	.023866
10	.028607		.027369		.025737	-.073584	.024105	.023101	.024856
11	.061464	-.072156	.069626		.075414	.085022	.081202	.091541	.081388
12	.000268	-.013665	.000256	-.002258	-.010181	.007811	.019514	.019226	.020311
13	.061464	.072156	.069626		.075414	.085022	.081202	.091541	.081388
14	.000268	-.042272	.000256	-.029627	.000241	-.017926	.000226	-.004879	.000224
15	.000268	-.013665	.000256	-.002258	-.010181	.007811	.019514	.019226	.020311
16	.000268	-.042272	.000256		.000241	-.017926	.000226	-.004879	.000224
17	.020188	.014942	.030136		.038279	.033548	.046422	.091541	.046952
18	.000268	-.070879	.000256		.000241	-.043663	.000226	-.028984	.000224
19	.000269	-.042272	.000257		.000242	-.017926	.000227	-.004879	.000225
20	.000269	-.071879	.000257		.000242	-.043663	.000227	-.028984	.000225
21	.558475	.141076	.545122		.522551	.046694	.501326	-.007936	.496362